

Braidings of Tensor Spaces

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Abstract

Let V be a braided vector space, that is, a vector space together with a solution $\hat{R} \in \text{End}(V \otimes V)$ of the Yang–Baxter equation. Denote $T(V) := \bigoplus_k V^{\otimes k}$. We associate to \hat{R} a solution $T(\hat{R}) \in \text{End}(T(V) \otimes T(V))$ of the Yang–Baxter equation on the tensor space $T(V)$. The correspondence $\hat{R} \rightsquigarrow T(\hat{R})$ is functorial with respect to V .

1 Introduction

Solutions of the Yang–Baxter equation find applications in statistical models, integrable systems, knot theory, representation theory of braid groups and many other areas. In the present article we propose a method for constructing new solutions of the Yang–Baxter equation. These new solutions live in the tensor space $T(V) := \bigoplus_k V^{\otimes k}$, where V is a vector space equipped with a solution \hat{R} of the Yang–Baxter equation (in other words, V is a *braided* vector space). The initial data for the new solution includes, in addition to the operator \hat{R} , a scalar parameter q .

We make a "block triangular" (in the sense specified in the section 3) Ansatz. It turns out that there is a "universal" system of equations in the algebra of the infinite Artin braid group (see section 2 for the definition of the braid group and connections between braid groups and braided vector spaces), solving which we automatically produce the solution of the Yang–Baxter equation on $T(V)$; we present a solution of this universal system. As a by-product of the universality, the correspondence $\hat{R} \rightsquigarrow T(\hat{R})$ is functorial in the following sense. Let V and V' be two braided vector spaces, with braidings \hat{R} and \hat{R}' respectively. Let $f \in \text{Hom}(V, V')$ be a homomorphism of vector spaces; it extends to the homomorphism $T(f) : T(V) \rightarrow T(V')$ by the natural rule: the restriction of $T(f)$ on $V^{\otimes k}$ is $f \otimes f \otimes \cdots \otimes f$ (k times). The functoriality means: if f intertwines \hat{R} and \hat{R}' , that is, $(f \otimes f)\hat{R} = \hat{R}'(f \otimes f)$, then $T(f)$ intertwines $T(\hat{R})$ and $T(\hat{R}')$.

Given a braiding on a vector space V , there is a well-known standard braiding on $T(V)$ which we call diagonal (see sections 2 and 3 for details). It should be noted that the braiding $T(\hat{R})$ is different from the standard diagonal braiding: it mixes tensors with different number of indices.

The solution of the universal system is formulated with the help of two kinds of elements in the braid group algebra: generalizations of the binomial coefficients, called *braid shuffle* elements, and the

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braid Pochhammer symbols; see section 2 for definitions. The braid shuffle elements and Pochhammer symbols satisfy certain identities, generalizing the corresponding classical identities. These generalized identities (some of them appear to be new and are given with a proof) are presented in section 2 as well. The solution itself is given in section 3; also, some additional properties of the solution $T(\hat{R})$ and some directions of future research are outlined there.

2 Notation, conventions and preliminary identities

Formulas hereafter often contain sums and products. It is always assumed that an empty sum (that is, $\sum_{j=a}^{a-1}$ for any integer a) is equal to zero and an empty product (that is, $\prod_{j=a}^{a-1}$ for any integer a) is equal to 1. The set of non-negative integers is denoted by $\mathbb{Z}_{\geq 0}$.

Braid group. We shall use the Artin presentation of the braid group B_{n+1} by generators σ_i , $1 \leq i \leq n$, and relations

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i - j| = 1, \quad (1)$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| > 1. \quad (2)$$

The first of these is called the *braid* relation and the second one the *far commutativity* (it reflects the commutativity of unlinked braids).

One has the *tower* $B_0 \subset B_1 \subset B_2 \subset \dots \subset B_\infty$ (the groups B_0 and B_1 are trivial) defined by inclusions $B_n \ni \sigma_i \mapsto \sigma_i \in B_{n+1}$, $i = 1, \dots, n-1$; the group $B_\infty = \varinjlim B_n$ is the inductive limit, the group with the countable set $\{\sigma_1, \sigma_2, \dots\}$ of generators subject to the above relations.

Given $\ell \in \mathbb{Z}_{\geq 0}$, we denote the endomorphism of B_∞ , sending σ_i to $\sigma_{i+\ell}$, $i = 1, 2, \dots$, by \uparrow^ℓ ,

$$\uparrow^\ell : \sigma_i \mapsto \sigma_{i+\ell}, \quad (3)$$

as in [12]. We write w^{\uparrow^ℓ} for the image of an element w ; for example, $\sigma_i^{\uparrow^\ell} = \sigma_{i+\ell}$.

Braided vector spaces. Let V a braided vector space of dimension N over a field \mathfrak{k} , with a braiding $\hat{R} \in \text{Aut}(V^{\otimes 2})$; \hat{R} is an invertible solution of the Yang–Baxter equation. The braiding induces the so-called *local* representation $\rho_{\hat{R}}$ of the braid group tower, $\rho_{\hat{R}} : B_n \rightarrow \text{Aut}(V^{\otimes n})$, by $\rho_{\hat{R}}(\sigma_i) = \hat{R}_i := \mathbb{1}_V^{\otimes(i-1)} \otimes \hat{R} \otimes \mathbb{1}_V^{\otimes(n-i-1)}$ (that is, σ_i acts as \hat{R} on the copies number i and $i+1$ of the space V in $V \otimes V \otimes V \dots$ and as the identity on the other copies). We use the same notation \uparrow^j for the shift in the copies of the space V ; for example, $\rho_{\hat{R}}(\sigma_i) = \rho_{\hat{R}}(\sigma_1)^{\uparrow^{i-1}}$.

In the sequel we slightly abuse the notation writing σ instead of $\rho_{\hat{R}}(\sigma)$; this should not produce any confusion: one can say that we consider $V \otimes V \otimes V \dots$ as a B_∞ -module, σ_i acts as \hat{R}_i . When needed we shall specify the Yang–Baxter operator \hat{R} defining the local representation.

We use the notation x_a , as in [6], for a vector in the copy number a of the space V ; $[a]$ is the set $\{1, 2, \dots, a\}$; we write $x_{[a]}$ for a tensor $x^{i_1 i_2 \dots i_a} \in V^{\otimes a}$ with a indices and $x_{[a]}^{\uparrow^b}$ for the same tensor x belonging to the copies numbered from $1+b$ to $a+b$; for example, $x_{[a]} y_{[b]}^{\uparrow^a}$ is an element $x^{i_1 i_2 \dots i_a} \otimes y^{j_1 j_2 \dots j_b} \in V^{\otimes a} \otimes V^{\otimes b}$.

The braiding can be understood as a rule of exchanging two copies of the space V ,

$$x_1 y_2 = \sigma_1 y_1 x_2. \quad (4)$$

Assume the same exchange rule for copies i and $i+1$ (i is arbitrary) of the space V , $x_i y_{i+1} = \sigma_i y_i x_{i+1}$; assume the same exchange rules for $x_i z_{i+1}$ and $y_i z_{i+1}$. The Yang-Baxter relation for \hat{R} (the braid relation for σ) ensures the coincidence of rewriting $x_1 y_2 z_3$ in two ways (starting with $x_1 y_2$ or $y_2 z_3$) to the form $z_1 y_2 x_3$.

The arrangement of the product $x_1 \dots x_k y_{k+1} \dots y_{k+l}$ to the form $y_1 \dots y_\ell x_{\ell+1} \dots x_{\ell+k}$ induces the exchange rule for decomposable tensors,

$$x_1 \rangle x_2 \rangle \dots x_k \rangle y_{k+1} \rangle y_{k+2} \rangle \dots y_{k+l} \rangle = \beta_{k,l} y_1 \rangle y_2 \rangle \dots y_\ell \rangle x_{\ell+1} \rangle x_{\ell+2} \rangle \dots x_{\ell+k} \rangle , \quad (5)$$

$\beta_{k,l} : V^{\otimes k} \otimes V^{\otimes l} \rightarrow V^{\otimes l} \otimes V^{\otimes k}$. For example, $\beta_{k,1} = \sigma_k \sigma_{k-1} \dots \sigma_1$ and $\beta_{1,l} = \sigma_1 \sigma_2 \dots \sigma_l$. The arrangement can be done, among other ways, by moving y 's one after another to the left or x 's to the right, giving two expressions for $\beta_{k,l}$,

$$\begin{aligned} \beta_{k,l} &= (\sigma_k \sigma_{k+1} \dots \sigma_{k+l-1}) (\sigma_{k-1} \sigma_k \dots \sigma_{k+l-2}) \dots (\sigma_1 \sigma_2 \dots \sigma_l) \\ &= (\sigma_k \sigma_{k-1} \dots \sigma_1) (\sigma_{k+1} \sigma_k \dots \sigma_2) \dots (\sigma_{k+l-1} \sigma_{k+l-2} \dots \sigma_l) . \end{aligned} \quad (6)$$

The equality of two expressions for β_{kl} in (6) is guaranteed by the Yang-Baxter equation for σ .

The identities

$$\beta_{l+m,n} = \beta_{m,n}^{\uparrow l} \beta_{l,n} \quad , \quad \beta_{l,m+n} = \beta_{l,m} \beta_{l,n}^{\uparrow m} \quad (7)$$

ensure the consistency of eq.(5) and it then follows that

$$\beta_{m,j} \beta_{m+l,n}^{\uparrow j} = \beta_{l,n}^{\uparrow m+j} \beta_{m,j+n} \quad , \quad \beta_{n,m+l}^{\uparrow j} \beta_{j,m} = \beta_{j+n,m} \beta_{n,l}^{\uparrow m+j} . \quad (8)$$

We shall often use the identity

$$\beta_{k,l} \phi \psi^{\uparrow l} = \psi \phi^{\uparrow k} \beta_{k,l} , \quad (9)$$

valid for arbitrary elements $\phi \in \mathfrak{k}B_l$ and $\psi \in \mathfrak{k}B_k$ of the group algebras.

Lift of the longest element. Denote by ω_a the lift of the longest element from the symmetric group S_a to B_a . It satisfies

$$\omega_a \phi = \phi' \omega_a \quad \forall \phi \in \mathfrak{k}B_a , \quad (10)$$

where $' : B_a \rightarrow B_a$ is the automorphism $\sigma_i \mapsto \sigma_{a-i}$ of B_a . We have

$$\omega_{a+b} = \beta_{a,b} \omega_a^{\uparrow b} \omega_b . \quad (11)$$

Shuffles. We shall use braid analogues of the $(q-)$ binomial coefficients, the braid shuffle elements $\text{III}_{m,n}$, indexed by two integers m, n with $m+n \in \mathbb{Z}_{\geq 0}$. The shuffle element $\text{III}_{m,n}$ belongs to the group ring $\mathfrak{k}B_{m+n}$ (and therefore to the inductive limit of the tower, $\mathfrak{k}B_\infty$); the shuffle elements are uniquely defined by the initial condition $\text{III}_{n,-n} = \delta_{n,0}$, $n \in \mathbb{Z}$, (δ is the Kronecker symbol) and any of the recurrence braid analogues of the Pascal rule (these two recurrences produce the same sets of shuffle elements)

$$\text{III}_{m,n} = \text{III}_{m-1,n} + \text{III}_{m,n-1} \beta_{m,1}^{\uparrow n-1} , \quad (12)$$

$$\text{III}_{m,n} = \text{III}_{m,n-1}^{\uparrow 1} + \text{III}_{m-1,n}^{\uparrow 1} \beta_{1,n} . \quad (13)$$

In particular, $\text{III}_{0,n} = 1$ and $\text{III}_{n,0} = 1$ for $n \in \mathbb{Z}_{\geq 0}$; for $i + j \in \mathbb{Z}_{\geq 0}$, $\text{III}_{i,j} = 0$ if $i < 0$ or $j < 0$. We have

$$\text{III}_{n+k,m} \text{III}_{k,n}^{\uparrow m} = \text{III}_{k,m+n} \text{III}_{n,m} . \quad (14)$$

The shuffle elements are denoted in several different ways in the literature. We use the notation of [7].

Braid Pochhammer symbols. Let

$$P_{k,n}(x, y) := (x - \beta_{k,1}y)(x - \beta_{k+1,1}y) \dots (x - \beta_{k+n-1,1}y) . \quad (15)$$

Here x and y are parameters. The elements $P_{k,n}(x, y)$ (defined for $k, n \in \mathbb{Z}_{\geq 0}$) can be understood as braid analogues of the $(q-)$ Pochhammer symbols.

By definition,

$$P_{k,n}(x, y) = P_{k,a}(x, y) P_{k+a,n-a}(x, y) . \quad (16)$$

Lemma. We have

$$P_{k,n}(x, z) = \sum_a \text{III}_{n-a,a}^{\uparrow k} \beta_{k,a} P_{0,a}(y, z) P_{k,n-a}(x, y)^{\uparrow a} . \quad (17)$$

Note. In (17) and in sums hereafter we often omit the range of summation: one may understand the range as \mathbb{Z} ; however the sums are finite because the summands differ from 0 only for a finite number of values; for instance, in (17) the shuffle element differs from zero for a from 0 to n only.

Proof. Induction in n . For $n = 0$ there is nothing to prove. Next,

$$\begin{aligned} P_{k,n+1}(x, z) &= P_{k,n}(x, z) \cdot (x - \beta_{k+n,1}y) \\ &= \sum_a \text{III}_{n-a,a}^{\uparrow k} \beta_{k,a} P_{0,a}(y, z) P_{k,n-a}(x, y)^{\uparrow a} \cdot (x - \beta_{k+n,1}y) \\ &= \sum_a \text{III}_{n-a,a}^{\uparrow k} \beta_{k,a} P_{0,a}(y, z) P_{k,n-a}(x, y)^{\uparrow a} \cdot \left((x - \beta_{k+n-a,1}^{\uparrow a}y) + \beta_{k+n-a,1}^{\uparrow a}(y - \beta_{a,1}z) \right) \\ &= \sum_a \text{III}_{n-a,a}^{\uparrow k} \left(\beta_{k,a} P_{0,a}(y, z) P_{k,n-a+1}(x, y)^{\uparrow a} + \beta_{k,a} \beta_{k+n-a,1}^{\uparrow a} P_{0,a+1}(y, z) P_{k,n-a}(x, y)^{\uparrow a+1} \right) \\ &= \sum_a (\text{III}_{n-a,a} + \text{III}_{n-a+1,a-1} \beta_{n-a+1,1}^{\uparrow a-1})^{\uparrow k} \beta_{k,a} P_{0,a}(y, z) P_{k,n-a+1}(x, y)^{\uparrow a} \\ &= \sum_a \text{III}_{n+1-a,a}^{\uparrow k} \beta_{k,a} P_{0,a}(y, z) P_{k,n-a+1}(x, y)^{\uparrow a} . \end{aligned}$$

In the fourth equality we used (16) to write $P_{k,n-a}(x, y)^{\uparrow a} (x - \beta_{k+n-a,1}^{\uparrow a}y) = P_{k,n+1-a}(x, y)^{\uparrow a}$; then we moved $\beta_{k+n-a,1}^{\uparrow a}$ (the factor in front of $(y - \beta_{a,1}z)$) to the left through $P_{0,a}(y, z) P_{k,n-a}(x, y)^{\uparrow a}$ and it became $P_{0,a}(y, z) P_{k,n-a}(x, y)^{\uparrow a+1}$ in view of (9) and far commutativity; then, $(y - \beta_{a,1}z)$ commutes with $P_{k,n-a}(x, y)^{\uparrow a+1}$ and $P_{0,a}(y, z)(y - \beta_{a,1}z) = P_{0,a+1}(y, z)$. In the fifth equality we separated the sums, used that $\beta_{k,a} \beta_{k+n-a,1}^{\uparrow a} = \beta_{n-a,1}^{\uparrow k+a} \beta_{k,a+1}$ by (8), shifted the summation index in the second sum and rewrote the result as a single sum. In the sixth equality we used (12). \square

Since $P_{i,j}(x, 0) = x^j$ and $P_{0,j}(0, z) = (-1)^j \omega_j z^j$, we find, evaluating (17) at $y = 0$, that

$$P_{k,n}(x, z) = \sum_a (-1)^a \text{III}_{n-a,a}^{\uparrow k} \beta_{k,a} \omega_a z^a x^{n-a} . \quad (18)$$

Braid Vandermonde identity. Substituting (18) into (16) and collecting the powers of x , we obtain, using (7)-(11) and far commutativity, the braid version of the Vandermonde identity

$$\mathbb{I}\mathbb{I}_{m+n-a,a}^{\uparrow k} \beta_{k,a} = \sum_{b,c:b+c=a} \mathbb{I}\mathbb{I}_{m-b,b}^{\uparrow k} \mathbb{I}\mathbb{I}_{n-c,c}^{\uparrow m+k} \beta_{k,b} \beta_{m+k-b,c}^{\uparrow b} \quad , \quad a, m, n, k \in \mathbb{Z}_{\geq 0} . \quad (19)$$

Since $\beta_{k,b} \beta_{m+k-b,c}^{\uparrow b} = \beta_{m-b,c}^{\uparrow b} \beta_{k,a}$ by (8), eq.(19) can be derived from its particular case $k = 0$,

$$\mathbb{I}\mathbb{I}_{m+n-a,a} = \sum_{b,c:b+c=a} \mathbb{I}\mathbb{I}_{m-b,b} \mathbb{I}\mathbb{I}_{n-c,c}^{\uparrow m} \beta_{m-b,c}^{\uparrow b} \quad , \quad a, m, n \in \mathbb{Z}_{\geq 0} . \quad (20)$$

Eq.(20) generalizes the defining recursion for the shuffle elements: setting n to 1, we reproduce (12); setting m to 1, we reproduce (13).

Another version of the Vandermonde identity is

$$\omega_j \mathbb{I}\mathbb{I}_{e,c}^{\uparrow j} = \sum_a (-1)^a \mathbb{I}\mathbb{I}_{j-a,a} \omega_{j-a}^{\uparrow a} \mathbb{I}\mathbb{I}_{e-a,c+j}^{\uparrow a} \beta_{a,c+j} \quad , \quad e, c, j \in \mathbb{Z}_{\geq 0} . \quad (21)$$

It is proved by induction on $e + c$. For $e = 0$ the relation (21) clearly holds. Thus, due to the initial conditions, it is enough to increase e , assuming that (21) holds for all smaller values of $e + c$. The combination $\mathcal{X}_{a,b;c} := \mathbb{I}\mathbb{I}_{a,b}^{\uparrow c} \beta_{c,b}$ (entering the right hand side of (21)) verifies the recursion $\mathcal{X}_{a+1,b;c} = \mathcal{X}_{a,b;c} + \mathcal{X}_{a+1,b-1;c} \beta_{a+c+1}^{\uparrow b-1}$ which straightforwardly implies the induction step.

Notes. 1. The lower labels of many (but not all) elements carry information about the number of strands needed to define these elements (and therefore the number of copies of the vector space in a local representation of the braid group tower). For example, $\mathbb{I}\mathbb{I}_{i,j}, P_{i,j}(x, y), \beta_{i,j} \in \mathfrak{k}B_{i+j}$ and $\omega_i \in \mathfrak{k}B_i$; this should not be confused with a different meaning of the lower label j of σ_j .

2. The case $k = 0$ of the (17), (18) and (19) was discussed in [1].

3 Braiding on the space of tensors

Extending the exchange rules (5) to all, not necessarily decomposable, tensors, we obtain the standard braiding

$$x_{[k]} y_{[l]}^{\uparrow k} = \beta_{k,l} y_{[l]} x_{[k]}^{\uparrow l} , \quad (22)$$

on the family $\{V^{\otimes k}\}_{k=0}^{\infty}$ of vector spaces (see, e.g., [10]). The corresponding braiding of the space $T(V)$ we call *diagonal*. The diagonal braiding of tensors is used, for example, in the construction of the q -Minkowski vectors as bi-spinors [13, 14] as well as in the q -versions of the accidental isomorphisms of semi-simple Lie groups [8].

We shall construct another natural braiding $T(\hat{R})$ on the tensor space $T(V) := \bigoplus_k V^{\otimes k}$. "Natural" here means (i) functorial with respect to V ; (ii) there is a quantum group associated to the braided space V (see, e.g. [3]). Its action naturally extends to $T(V)$. The braiding $T(\hat{R})$ is covariant with respect to this quantum group.

Viewed differently, the above covariance reflects a particular property of our solution: the building blocks of the braiding $T(\hat{R})$ are certain polynomials in the original R -matrix; the Yang-Baxter system

of equations can be formulated on the universal level of the braid group. Our solution belongs to the ring of the monoid of positive braids. Questions related to the uniqueness of our solution will be discussed in [4]. A version of the braiding on the tensor space in which the building blocks of $T(\hat{R})$ are filled modulo a given natural number N will be considered in [5].

Due to the canonical isomorphisms $V^{\otimes l} \otimes V^{\otimes k} \rightarrow V^{\otimes(l+k)}$, we can consider a more general Ansatz for a braiding on $T(V)$, for which the right hand side of (22) acquires other terms $y_{[l']}\overset{\uparrow}{x}_{[k']}$ with $k' + l' = k + l$. Our aim is to study such braidings.

Denote by $\left\{ \begin{smallmatrix} a & b \\ c & \end{smallmatrix} \right\}$ the submatrix appearing in front of a term $y_{[k]}\overset{\uparrow}{x}_{[b+c-k]}$. Thus the braiding $T(\hat{R})$, understood as the exchange rule, reads

$$x_{[b]}\overset{\uparrow}{y}_{[c]} = \sum_{k=0}^{b+c} \left\{ \begin{smallmatrix} b & c \\ k & \end{smallmatrix} \right\} y_{[k]}\overset{\uparrow}{x}_{[b+c-k]}. \quad (23)$$

The full form of $\left\{ \begin{smallmatrix} b & c \\ k & \end{smallmatrix} \right\} y_{[k]}\overset{\uparrow}{x}_{[b+c-k]}$ is $\left\{ \begin{smallmatrix} b & c \\ k & \end{smallmatrix} \right\}_{j_1 \dots j_{b+c}}^{i_1 \dots i_{b+c}} y^{j_1 \dots j_k} x^{j_{k+1} \dots j_{b+c}}$, the summation in repeated indices is assumed.

As for the standard (Drinfeld–Jimbo, [2, 9]) constant R -matrices, it turns out that a braiding with all $\left\{ \begin{smallmatrix} b & c \\ k & \end{smallmatrix} \right\}$ non-vanishing does not exist. We require the R -matrix for $T(V)$ to be block-triangular (like in the construction of the orthogonal and symplectic R -matrices in [11]). In other words we restrict ourselves to braidings with $\left\{ \begin{smallmatrix} b & c \\ k & \end{smallmatrix} \right\} = 0$ if $k > c$. Then the summation in (23) shortens to

$$x_{[b]}\overset{\uparrow}{y}_{[c]} = \sum_{k=0}^c \left\{ \begin{smallmatrix} b & c \\ c-k & \end{smallmatrix} \right\} y_{[c-k]}\overset{\uparrow}{x}_{[b+k]}. \quad (24)$$

For example,

$$x_{[3]}\overset{\uparrow}{y}_{[2]} = \left\{ \begin{smallmatrix} 3 & 2 \\ 2 & \end{smallmatrix} \right\} y_{[2]}\overset{\uparrow}{x}_{[3]} + \left\{ \begin{smallmatrix} 3 & 2 \\ 1 & \end{smallmatrix} \right\} y_{[1]}\overset{\uparrow}{x}_{[4]} + \left\{ \begin{smallmatrix} 3 & 2 \\ 0 & \end{smallmatrix} \right\} y_{[0]}\overset{\uparrow}{x}_{[5]}.$$

The operation $\overset{\uparrow}{}$ is the identity and can be omitted, $x_{[5]}^{\uparrow 0} = x_{[5]}$.

The submatrix $\left\{ \begin{smallmatrix} 1 & 1 \\ 1 & \end{smallmatrix} \right\}$ is an endomorphism of $V \otimes V$. With our triangular Ansatz, this endomorphism must verify the Yang-Baxter equation. We identify $\left\{ \begin{smallmatrix} 1 & 1 \\ 1 & \end{smallmatrix} \right\}$ with the initial braiding on V . We shall see that our solution contains two "parameters", the Yang-Baxter operator $\hat{R} = \left\{ \begin{smallmatrix} 1 & 1 \\ 1 & \end{smallmatrix} \right\}$ and the scalar $q = \left\{ \begin{smallmatrix} 0 & 0 \\ 0 & \end{smallmatrix} \right\}$ (by convention, $x_{[0]} \in V^{\otimes 0} \simeq \mathfrak{k}$).

To be a braiding, $T(\hat{R})$ must satisfy the Yang-Baxter equation. As for vectors, we shall understand the Yang–Baxter equation for an operator \hat{R} as the equality of two different reorderings of $x^\bullet y^\bullet z^\bullet$ (using the rule (24) for $x^\bullet y^\bullet$, $x^\bullet z^\bullet$ and $y^\bullet z^\bullet$) to the form $z^\bullet y^\bullet x^\bullet$.

We have

$$x_{[a]}\overset{\uparrow}{y}_{[b]}z_{[c]}^{\uparrow a+b} = \sum_{d,e,f} \left\{ \begin{smallmatrix} b & c \\ d & \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} a & d \\ e & \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} a+d-e & b+c-d \\ f & \end{smallmatrix} \right\} z_{[e]}^{\uparrow e} y_{[f]}^{\uparrow e} x_{[a+b+c-e-f]}^{\uparrow e+f} \quad (25)$$

and

$$(x_{[a]}\overset{\uparrow}{y}_{[b]})z_{[c]}^{\uparrow a+b} = \sum_{u,v,w} \left\{ \begin{smallmatrix} a & b \\ u & \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} a+b-u & c \\ v & \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} u & v \\ w & \end{smallmatrix} \right\} z_{[w]}^{\uparrow w} y_{[u+v-w]}^{\uparrow w} x_{[a+b+c-u-v]}^{\uparrow u+v}. \quad (26)$$

Equating terms, we arrive at a system

$$\sum_i \left\{ \begin{smallmatrix} b & c \\ i & \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} a & i \\ e & \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} a+i-e & b+c-i \\ f & \end{smallmatrix} \right\}^{\uparrow e} = \sum_j \left\{ \begin{smallmatrix} a & b \\ j & \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} a+b-j & c \\ f+e-j & \end{smallmatrix} \right\}^{\uparrow j} \left\{ \begin{smallmatrix} j & f+e-j \\ e & \end{smallmatrix} \right\}. \quad (27)$$

Here the summation is over the terms which are non zero. This implies that $a, b, c, e, f \in \mathbb{Z}_{\geq 0}$; moreover, in the left hand side the summation is over i such that $\max(0, e) \leq i \leq \min(c, b + c - f)$ and, in the right hand side, the summation is over j such that $\max(0, f + e - c) \leq j \leq \min(f, b)$.

The system (27) can be considered on the "universal" level, as a system of equations in the braid group ring. We shall take this point of view and construct a universal solution of the system (27), with $\{a \ b\} \in \mathfrak{k}B_{a+b}$. For a given braiding \hat{R} on the vector space V , the braiding $T(\hat{R})$ on $T(V)$ is obtained by taking the local representation $\rho_{q\hat{R}}$ of the braid group tower ($\rho_{q\hat{R}}$ sends σ_i to $q\hat{R}_i$).

Theorem. Let $p_{a,b} := P_{a,b}(1, q^{-2})$. The following elements

$$\{a \ b\} = q^{1-a-c} \text{III}_{b-c,c}^{\uparrow a} \beta_{a,c} p_{a,b-c}^{\uparrow c}, \quad (28)$$

provide a universal solution of the system (27).

Proof. The left hand side of (27) multiplied by $q^{2a+b+f-3}$ reads:

$$\sum_i q^{-2i} \text{III}_{c-i,i}^{\uparrow a+b} \beta_{b,i}^{\uparrow a} p_{b,c-i}^{\uparrow a+i} \text{III}_{i-e,e}^{\uparrow a} \beta_{a,e} p_{a,i-e}^{\uparrow e} \text{III}_{b+c-f-i,f}^{\uparrow a+i} \beta_{a+i-e,f}^{\uparrow e} p_{a+i-e,b+c-f-i}^{\uparrow e+f}.$$

The term $p_{a,i-e}^{\uparrow e}$ commutes with $\text{III}_{b+c-f-i,f}^{\uparrow a+i}$ (far commutativity). When $p_{a,i-e}^{\uparrow e}$ moves to the right through $\beta_{a+i-e,f}^{\uparrow e}$ it becomes $p_{a,i-e}^{\uparrow e+f}$ by (9). Next, by (16), $p_{a,i-e}^{\uparrow e+f} p_{a+i-e,b+c-f-i}^{\uparrow e+f} = p_{a,b+c-f-e}^{\uparrow e+f}$ and it does not depend on the summation index. After, the term $\text{III}_{i-e,e}^{\uparrow a}$ commutes with $p_{b,c-i}^{\uparrow a+i}$ (far commutativity); then $\text{III}_{i-e,e}^{\uparrow a}$ moves to the left through $\beta_{b,i}^{\uparrow a}$ and becomes $\text{III}_{i-e,e}^{\uparrow a+b}$ by (9); by (14), we have $\text{III}_{c-i,i}^{\uparrow a+b} \text{III}_{i-e,e}^{\uparrow a+b} = \text{III}_{c-e,e}^{\uparrow a+b} \text{III}_{c-i,i-e}^{\uparrow a+b+e}$ and the left hand side of (27) becomes

$$\text{III}_{c-e,e}^{\uparrow a+b} \left(\sum_i q^{-2i} \text{III}_{c-i,i-e}^{\uparrow a+b+e} \beta_{b,i}^{\uparrow a} p_{b,c-i}^{\uparrow a+i} \beta_{a,e} \text{III}_{b+c-f-i,f}^{\uparrow a+i} \beta_{a+i-e,f}^{\uparrow e} \right) p_{a,b+c-f-e}^{\uparrow e+f}. \quad (29)$$

The term $\beta_{a,e}$ commutes with $\text{III}_{b+c-f-i,f}^{\uparrow a+i}$ (far commutativity). By (8), $\beta_{a,e} \beta_{a+i-e,f}^{\uparrow e} = \beta_{i-e,f}^{\uparrow e+a} \beta_{a,e+f}$; the factor $\beta_{a,e+f}$ does not depend on the summation index and goes out from the sum. By (7), $\beta_{b,i}^{\uparrow a} = \beta_{b,e}^{\uparrow a} \beta_{b,i-e}^{\uparrow a+e}$. The factor $\beta_{b,e}^{\uparrow a}$ does not depend on the summation index, it commutes with $\text{III}_{c-i,i-e}^{\uparrow a+b}$ (far commutativity) and moves out from the sum to the left. The left hand side of (27) takes now the form

$$\text{III}_{c-e,e}^{\uparrow a+b} \beta_{b,e}^{\uparrow a} \left(\sum_i q^{-2i} \text{III}_{c-i,i-e}^{\uparrow a+b+e} \beta_{b,i-e}^{\uparrow e} p_{b,c-i}^{\uparrow i} \text{III}_{b+c-f-i,f}^{\uparrow i} \beta_{i-e,f}^{\uparrow e} \right)^{\uparrow a} \beta_{a,e+f} p_{a,b+c-f-e}^{\uparrow e+f}. \quad (30)$$

The right hand side of (27) multiplied by $q^{2a+b+f-3}$ reads:

$$q^{-2e} \sum_j \text{III}_{b-j,j}^{\uparrow a} \beta_{a,j} p_{a,b-j}^{\uparrow j} \text{III}_{c-e-f+j,e+f-j}^{\uparrow a+b} \beta_{a+b-j,f+e-j}^{\uparrow j} p_{a+b-j,c-e-f+j}^{\uparrow e+f} \text{III}_{f-j,e}^{\uparrow j} \beta_{j,e} p_{j,f-j}^{\uparrow e}.$$

The term $p_{a,b-j}^{\uparrow j}$ commutes with $\text{III}_{c-e-f+j,e+f-j}^{\uparrow a+b}$ (far commutativity); when $p_{a,b-j}^{\uparrow j}$ moves to the right through $\beta_{a+b-j,f+e-j}^{\uparrow j}$, it becomes $p_{a,b-j}^{\uparrow e+f}$ by (9); then, by (16) we have $p_{a,b-j}^{\uparrow e+f} p_{a+b-j,c-e-f+j}^{\uparrow e+f} = p_{a,b+c-e-f}^{\uparrow e+f}$; the term $p_{a,b+c-e-f}^{\uparrow e+f}$ moves to the right out of the sum without changes (far commutativity). Next, the term $\text{III}_{f-j,e}^{\uparrow j}$ moves to the left through $\beta_{a+b-j,f+e-j}^{\uparrow j}$, becoming $\text{III}_{f-j,e}^{\uparrow a+b}$ by (9); we have

$\mathbb{I}_{c-e-f+j,e+f-j}^{\uparrow a+b} \mathbb{I}_{f-j,e}^{\uparrow a+b} = \mathbb{I}_{c-e,e}^{\uparrow a+b} \mathbb{I}_{c-e-f+j,f-j}^{\uparrow a+b+e}$ by (14); the term $\mathbb{I}_{c-e,e}^{\uparrow a+b}$ moves now without changes to the very left (far commutativity) and the right hand side becomes

$$\mathbb{I}_{c-e,e}^{\uparrow a+b} \left(q^{-2e} \sum_j \mathbb{I}_{b-j,j}^{\uparrow a} \beta_{a,j} \mathbb{I}_{c-e-f+j,f-j}^{\uparrow a+b+e} \beta_{a+b-j,f+e-j}^{\uparrow j} \beta_{j,e} p_{j,f-j}^{\uparrow e} \right) p_{a,b+c-f-e}^{\uparrow e+f}. \quad (31)$$

The term $\beta_{a,j}$ commutes with $\mathbb{I}_{c-e-f+j,f-j}^{\uparrow a+b+e}$ (far commutativity); then, by (8), $\beta_{a,j} \beta_{a+b-j,f+e-j}^{\uparrow j} = \beta_{b-j,e+f-j}^{\uparrow a+j} \beta_{a,e+f}$; the term $\beta_{a,e+f}$ does not depend on the summation index; we move it out of the sum to the right; when it moves through $\beta_{j,e} p_{j,f-j}^{\uparrow e}$, this expression transforms into $\beta_{j,e}^{\uparrow a} p_{j,f-j}^{\uparrow a+e}$. We then rewrite: $\beta_{b-j,e+f-j}^{\uparrow a+j} \beta_{j,e}^{\uparrow a} = \beta_{b,e}^{\uparrow a} \beta_{b-j,f-j}^{\uparrow a+j+e}$ by (8) and move the term $\beta_{b,e}^{\uparrow a}$ out of the sum to the left; it commutes with $\mathbb{I}_{c-e-f+j,f-j}^{\uparrow a+b+e}$ (far commutativity) and transforms $\mathbb{I}_{b-j,j}^{\uparrow a}$ into $\mathbb{I}_{b-j,j}^{\uparrow a+e}$ by (9). The right hand side of (27) takes the form

$$\mathbb{I}_{c-e,e}^{\uparrow a+b} \beta_{b,e}^{\uparrow a} \left(q^{-2e} \sum_j \mathbb{I}_{b-j,j}^{\uparrow e} \mathbb{I}_{c-e-f+j,f-j}^{\uparrow b+e} \beta_{b-j,f-j}^{\uparrow j+e} p_{j,f-j}^{\uparrow e} \right)^{\uparrow a} \beta_{a,e+f} p_{a,b+c-f-e}^{\uparrow e+f}. \quad (32)$$

Comparing (30) with (32), we see that the theorem will follow from the equality of

$$\sum_i q^{-2i} \mathbb{I}_{c-i,i-e}^{\uparrow b} \beta_{b,i-e} p_{b,c-i}^{\uparrow i-e} \mathbb{I}_{b+c-f-i,f}^{\uparrow i-e} \beta_{i-e,f} = q^{-2e} \sum_j \mathbb{I}_{b-j,j} \mathbb{I}_{c-e-f+j,f-j}^{\uparrow b} \beta_{b-j,f-j}^{\uparrow j} p_{j,f-j}. \quad (33)$$

Substitute (18) in the left hand side of (33):

$$\sum_{i,t} (-1)^t q^{-2i-2t} \mathbb{I}_{c-i,i-e}^{\uparrow b} \beta_{b,i-e} \mathbb{I}_{c-i-t,t}^{\uparrow b+i-e} \beta_{b,t}^{\uparrow i-e} \omega_t^{\uparrow i-e} \mathbb{I}_{b+c-f-i,f}^{\uparrow i-e} \beta_{i-e,f}. \quad (34)$$

Move $\beta_{b,i-e}$ to the right through $\mathbb{I}_{c-i-t,t}^{\uparrow b+i-e}$ (far commutativity); use (7) to write $\beta_{b,i-e} \beta_{b,t}^{\uparrow i-e} = \beta_{b,i+t-e}$ and (14) to write $\mathbb{I}_{c-i,i-e}^{\uparrow b} \mathbb{I}_{c-i-t,t}^{\uparrow b+i-e} = \mathbb{I}_{c-i-t,i+t-e}^{\uparrow b} \mathbb{I}_{t,i-e}^{\uparrow b}$; the term $\mathbb{I}_{t,i-e}^{\uparrow b}$ moves now to the right through $\beta_{b,i+t-e}$ becoming $\mathbb{I}_{t,i-e}$. Replace the summation index t by $r = i + t$:

$$\sum_{i,r} (-1)^{r+e} q^{-2r} \mathbb{I}_{c-r,r-e}^{\uparrow b} \beta_{b,r-e} \left\{ (-1)^{i-e} \mathbb{I}_{r-i,i-e} \omega_{r-i}^{\uparrow i-e} \mathbb{I}_{b+c-f-i,f}^{\uparrow i-e} \beta_{i-e,f} \right\}. \quad (35)$$

The sum over i (terms in braces) is taken with the help of (21) and the final form of the left hand side of (33) reads

$$\sum_r (-1)^{r+e} q^{-2r} \mathbb{I}_{c-r,r-e}^{\uparrow b} \beta_{b,r-e} \omega_{r-e} \mathbb{I}_{b+c-e-f,e+f-r}^{\uparrow r-e}. \quad (36)$$

Substitute (18) in the right hand side of (33):

$$q^{-2e} \sum_{j,s} (-1)^s q^{-2s} \mathbb{I}_{b-j,j} \mathbb{I}_{c-e-f+j,f-j}^{\uparrow b} \beta_{b-j,f-j}^{\uparrow j} \mathbb{I}_{f-j-s,s}^{\uparrow j} \beta_{j,s} \omega_s. \quad (37)$$

The element $\mathbb{I}_{f-j-s,s}^{\uparrow j}$ moves to the left through $\beta_{b-j,f-j}^{\uparrow j}$ becoming $\mathbb{I}_{f-j-s,s}^{\uparrow b}$ by (9). Then we write $\mathbb{I}_{c-e-f+j,f-j}^{\uparrow b} \mathbb{I}_{f-j-s,s}^{\uparrow b} = \mathbb{I}_{c-e-s,s}^{\uparrow b} \mathbb{I}_{c-e-f+j,f-j-s}^{\uparrow b+s}$ by (14) and $\beta_{b-j,f-j}^{\uparrow j} \beta_{j,s} = \beta_{b,s} \beta_{b-j,f-j-s}^{\uparrow j+s}$ by (8).

The shuffle element $\text{III}_{c-e-s,s}^{\uparrow b}$ moves through $\text{III}_{b-j,j}$ (far commutativity) to the left; the term $\beta_{b,s}$ moves to the left through the product of two shuffles, the shuffle $\text{III}_{c-e-f+j,f-j-s}^{\uparrow b+s}$ does not change (far commutativity) while the shuffle $\text{III}_{b-j,j}$ becomes $\text{III}_{b-j,j}^{\uparrow s}$. The right hand side of (33) reads now

$$q^{-2e} \sum_{j,s} (-1)^s q^{-2s} \text{III}_{c-e-s,s}^{\uparrow b} \beta_{b,s} \left\{ \text{III}_{b-j,j}^{\uparrow s} \text{III}_{c-e-f+j,f-j-s}^{\uparrow b+s} \beta_{b-j,f-j-s}^{\uparrow j+s} \right\} \omega_s . \quad (38)$$

The sum in j (terms in braces) is taken with the help of (19), it equals $\text{III}_{b+c-e-f,f-s}^{\uparrow s}$, which far commutes with ω_s and we arrive at the same expression (36). The proof is completed. \square

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